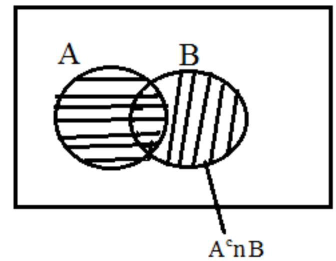
RANDOM EXPERIMENT :

An experiment that can be done any number of times under identical conditions in which
(i) outcomes of experiment are known.
(ii) outcomes of a particular trial is not known (each performance of a random experiment is called a trial)
Example :Tossing a coin, Throwing a dice ……



ELEMENTARY OR SIMPLE EVENT:

Result of trial in a random experiment is called Elementary event.

SAMPLE SPACE :

Set of all possible elementary events (outcomes) in a random experiment is called sample space.
For the random experiment – tossing a coin
$S = \{T, H\}$
For the random experiment – throwing a die
$S = \{1, 2, 3, 4, 5, 6\}$

EVENT:

Every non empty subset of a sample space of random experiment is called an event.

FAVOURABLE CASES:

Number of cases favourable to an event in trial is the number of outcomes which entail the happening of an event.
Example : Getting even number in experiment of throwing a die - Favourable cases are 3.

MUTUALLY EXCLUSIVE EVENTS:

Events are said to be mutually exclusive happening of one event, prevent the happening of other (or) A and B are any two events, if $A \cap B = \emptyset$ then A, B are mutually exclusive.
Example: 1. In the experiment of tossing a coin event of getting head and event of getting tail are mutually exclusive.
2. In the experiment of throwing a die, event of getting an even number, event of getting an odd number are mutually exclusive events.

EXHAUSTIVE EVENTS:

Events which consists of all possible outcomes in a random experiment are called exhaustive events (or) A, B are any two events if $A \cup B = S$ then A, B are called exhaustive events.
Example: In the experiment of throwing a die events
1. $A = \{1, 2, 3\}$, $B = \{3, 4, 5, 6\}$ are exhaustive events
2. $A = \{2, 4, 6\}, B = \{1, 3, 5\}$ are mutually exclusive and exhaustive events.

EQUALLY LIKELY EVENTS:

Outcomes of a trial are said to be equally likely if one cannot be expected in preference to the others
1. In throwing a die, all possible cases are equally likely
2. In tossing a coin, two possible cases are equally likely

PROBABILITY:

CLASSICAL DEFINITION OF PROBABILITY:

Random experiment contains n elementary events, m of them are favourable then ratio of favourable to total number of events is called Probability.

$P(E) = \frac{Favourable\ events}{Total\ no.of\ events} = \frac{n(E)}{n(S)}$

SURE EVENT:

An event which always happen is called Sure event.

Example: Event of getting a Tail or Head is a Sure event.

For Sure event $P(E) = 1$.

IMPOSSIBLE EVENT:

An event which never happen is called Impossible event.

Example : Event of getting a head and tail in tossing a coin is Impossible event

For impossible event $P(E) = 0 \quad \therefore 0 \le P(E) \le 1$

COMPLEMENTARY EVENT:

Two events of a sample space whose intersection is$\emptyset$ and whose union is sample space are called complementary events. If E is an event its complement is denoted by E^c.

$P(E^c) = \frac{n-m}{n} = 1 - \frac{m}{n}$ (If an experiment contains n events in which m are favourable)

$P(E^c) = 1 - P(E) \Rightarrow P(E) + P(E^c) = 1$.

PROBABILITY FUNCTION:

Let S be the sample space of a random experiment then the function $P: P(S) \to R$ satisfying following axioms is called Probability function.

$P(E) \ge 0; \ P(S) = 1$

THE AXIOMS OF PROBABILITY:

The axioms of probability are:

1. $0 \le P(A) \le 1\ for\ each\ event\ A \subseteq S$
2. $P(S) = 1$
3. If A and B are any two mutually exclusive events then $P(A \cup B) = P(A) + P(B)$

SOME ELEMENTARY THEOREMS :

* Probability of complementary event $P(A^c) = 1 - P(A)$

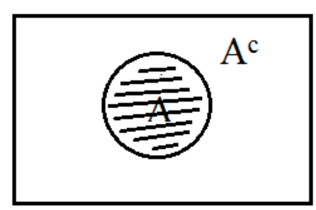
Proof: $S = A \cup A^c$

A and A^c are mutually exclusive events or A and A^c are disjoint sets

$\Rightarrow A \cap A^c = \emptyset$

$P(S) = P(A) + P(A^c)$

$= P(S) = 1 \Rightarrow P(A^c) = 1 - P(A)$



* For any events A and B $P(A^c \cap B) = P(B) - P(A \cap B)$

Proof: A∩B is shown in the horizontal lines

$A^c \cap B$by vertical lines

$A \cap B\ and\ A^c \cap B$are disjoint sets

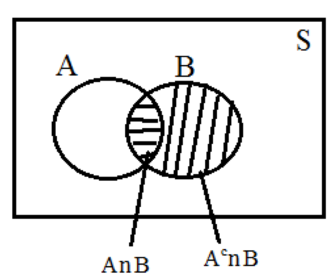
$\therefore A \cap B\ and\ A^c \cap B$ are mutually exclusive

$\therefore P[(A \cap B) \cup (A^c \cap B)] = P(A \cap B) + P(A^c \cap B)$ [from Axiom III]

But $[(A \cap B) \cup (A^c \cap B)] = B$

$\therefore P(B) = P(A \cap B) + P(A^c \cap B)$

$\Rightarrow P(A^c \cap B) = P(B) - P(A \cap B)$



Note :$P(A \cap B^c) = P(A) - P(A \cap B)$

ADDITION THEOREM IN PROBABILITY:

* For any two events A and B $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: $A \cap B \neq \emptyset$

Set A is shown by horizontal lines

$A^c \cap B$by vertical lines

$A\ and\ A^c \cap B$are disjoint sets

$A\ and\ A^c \cap B$ are mutually exclusive

$P\big(A \cup (A^c \cap B)\big) = P(A) + P(A^c \cap B)$

But $A \cup (A^c \cap B) = A \cup B$

$\Rightarrow P(A \cup B) = P\big(A \cup (A^c \cap B)\big) = P(A) + P(A^c \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note: $A \subseteq B$

i) $P(A^c \cap B) = P(B) - P(A)$

ii) $P(A) \leq P(B)$

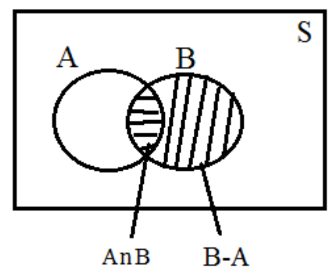
Second Method:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Case i) :Suppose $(A \cap B) = \emptyset\ then\ P(A \cap B) = 0$

$\Rightarrow P(A \cup B) = P(A) + P(B)$

Case ii) :Suppose $(A \cap B) \neq \emptyset\ then\ (A \cup B)$



$= A \cup (B - A) and\ A \cap (B - A) = \emptyset$

$P(A \cup B) = P\big(A \cup (B - A)\big) = P(A) + P(B - A)$

$= P(A) + P(B - (A \cap B))$

$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

$[E_1 \subseteq E_2\ then\ P(E_2 - E_1) = P(E_2) - P(E_1) \because [A \cap B \subset B]]$

THEOREM :

* For any three events A, B, C

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Proof: Write $B \cup C = D$

$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D)$

$= P(A) + P(B \cup C) - P(A \cap D)$

$= P(A) + P(B) + P(C) - P(B \cap C) - P\big(A \cap (B \cup C)\big)$

$= P(A) + P(B) + P(C) - P(B \cap C) - P\big((A \cap B) \cup (A \cap C)\big)$

$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)]$

$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

PROBLEMS:

1. Find the probability of getting one head in tossing two coins

Sol: Let A be the event of getting one head

$A = \{HT, TH\}$

$S = \{HH, HT, TH, TT\}$

Number of elements in S = 4 = n(S)

Number of elements in A = 2 = n(A)

$P(A) = \frac{2}{4} = 1/2 = \frac{n(A)}{n(S)}$

2. If three coins are tossed, find probability of getting 1) three heads, 2) two heads, 3) no heads

Sol: 1) Let A be the event of getting three heads
$A = \{HHH\}$
$S = \{HHH, HHT, THH, HTH, HTT, THT, TTH, TTT\}$
$P(A) = 1/8$
2) Let B be the event of getting two heads
$B = \{HHT, THH, HTH\}$
$P(B) = 3/8$
3) Let C be the event of getting no heads
$C = \{TTT\}$
$P(C) = 1/8$

3. Find the probability of getting sum 9 if two dice are thrown

Sol: Sample space $S = 6^2 = 36$
Let A be the event of getting a sum of 9
$A = \{(3,6), (4,5), (5,4), (6,3)\}$
$P(A) = \frac{4}{36} = \frac{1}{9}$

4. Find the probability of getting sum of 10 if we throw two dice

Sol: Given that 2 dice are thrown.
$\therefore n(S) = 6^2 = 36$
Let A be the event of getting a sum of 10
$A = \{(4,6), (5,5), (6,4)\}$
$P(A) = \frac{3}{36} = \frac{1}{12}$

5. When a dice is thrown, find the probability of getting i) even number and ii) odd numbers.

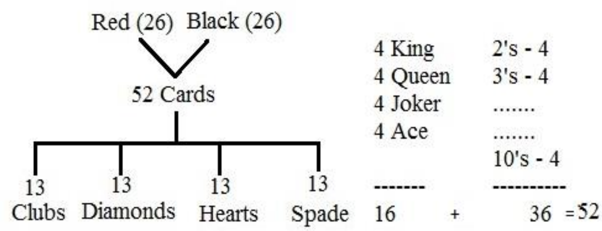
Sol: When a dice is thrown n(S) = 6
i) Let A be the event of getting an even number then n(A) =3, P(A) = 3/6 = ½.
ii) Let B be the event of getting an odd number then n(B) = 3, P(B) = 3/6 = ½.

6. A and B throw alternatively with a pair of dice, one who first throws a total of nine wins. What are their respective chances of winning if A starts the game.

Sol: When a pair of dice are thrown
Sample space $S = \{(1,1) \ldots\ldots (1,6)$
(2,1)……(2,6)
……………..
……………..
(6,1)…….(6,6)}
i.e., $n(S) = 36$
One who throws sum 9 wins
Favourable cases for this event are {(3,6), (4,5), (5, 4), (6, 3)}
i. e., number of favourable cases = 4
Probability of A throws 9 is $P(A) = \frac{4}{36} = \frac{1}{9}$
Probability of A not winning is $P(\bar{A}) = 1 - \frac{1}{9} = \frac{8}{9}$
For B also probability to get 9 is $P(B) = \frac{1}{9}$
Probability of not getting 9 is $P(\bar{B}) = \frac{8}{9}$

Probability of A's winning when A starts
$= P(A) + P(\bar{A}).P(\bar{B})P(A) + P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A) + \cdots ..$
$= \frac{1}{9} + \frac{8}{9}.\frac{8}{9}.\frac{1}{9} + \frac{8}{9}.\frac{8}{9}.\frac{8}{9}.\frac{8}{9}.\frac{1}{9} \ldots\ldots \quad = \frac{1}{9}\left[1 + \frac{8}{9}.\frac{8}{9} + \frac{8}{9}.\frac{8}{9}.\frac{8}{9}.\frac{8}{9} + \cdots .\right]$
$= \frac{1}{9}\left[1 + \left(\frac{8}{9}\right)^2 + \left(\left(\frac{8}{9}\right)^2\right)^2 + \cdots .\right] = \frac{1}{9}\left[\frac{1}{1-\left(\frac{8}{9}\right)^2}\right] = \frac{1}{9} \times \frac{81}{17} = \frac{9}{17} = 0.529 \quad \left[\because G.P\ S_\infty = \frac{a}{1-r}\right]$
Probability of A's winning $= 0.529$
Probability of B's winning $= 1 - 0.529 = 0.479$.

7. Find the probability of getting Red king if we select a card from a pack of 52 cards.
Sol: No. of cards $n(S) = 52$
Favourable cards two red kings = 2
i.e., probability $= \frac{2}{52} = \frac{1}{26}$



8. Find the probability of getting 2 diamonds random from a pack of 52 cards.
Sol: Selecting two cards from 52 $= {}^{52}C_2$
Favourable events selecting 2 diamonds from 13 is ${}^{13}C_2$
∴Required probability $= \frac{{}^{13}C_2}{{}^{52}C_2} = \frac{13\times12}{52\times51} = \frac{3}{51} = \frac{1}{17}$

9. One card is drawn from a regular deck of 52 cards. What is the probability of a card being either red or a king.
Sol: $n(S) = 52$
Let E_1 be the event of getting red
$n(E_1) = 26$
Let E_2 be the event of getting a king
$n(E_2) = 4$
$P(E_1 \cap E_2) = \frac{2}{52}$
$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{14}{26} = \frac{7}{13}$

10. Two cards are selected randomly from 10 cards numbered 1 to 10. Find the probability that the sum is even if
i) the two cards drawn together
ii) two cards are drawn one after another with replacement
Sol: i) Selecting two cards from 10 ${}^{10}C_2 = 45$
Sum is even if both are odd or both are even.
There are 5 even and 5 odd
Both are even cases ${}^{5}C_2 = 10$
Both are odd cases ${}^{5}C_2 = 10$
Required favourable cases$= 10 + 10 = 20$
Probability of selecting two cards $= \frac{20}{45} = \frac{4}{9}$

ii) Two cards are drawn one after another with replacement
No. of cases $= 10 \times 10 = 100$

No. of possible cases both are even $= 5 \times 5 = 25$
No. of possible cases both are odd $= 5 \times 5 = 25$
Total No. of possible cases $= 25 + 25 = 50$
Therefore required probability $= \frac{50}{100} = \frac{1}{2}$

11. Out of 15 bulbs, 4 are defective, 4 are selected at random. Find the probability that
i) All are defective bulbs ii) Two are defective bulbs

Sol: i) All are defective bulbs
Total No. of bulbs $= 15$
No. of defective bulbs $= 4$
A be the event of selecting all defective bulbs $n(A) = {}^4C_4 = 1$
No. of ways of selecting 4 bulbs from 15 $= {}^{15}C_4 = n(S)$
$\Rightarrow P(A) = \frac{{}^4C_4}{{}^{15}C_4} = \frac{1}{1365}$

ii) Two are defective bulbs
B be the event of getting two defective bulbs
$n(B) = {}^4C_2$; $P(B) = \frac{{}^4C_2}{{}^{15}C_2} = \frac{6}{1365}$

12. $P(A) = \frac{1}{5},\ P(B) = \frac{2}{3},\ P(A \cap B) = \frac{1}{15},$
$Find\ P(A \cup B), P(A^c \cap B), P(A \cap B^c),\ P(A^c \cap B^c),\ P(A^c \cup B^c)$

Sol: $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{5} + \frac{2}{3} - \frac{1}{15} = \frac{4}{5}$
$P(A^c \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{15} = \frac{9}{15} = \frac{3}{5}$
$P(A \cap B^c) = P(A) - P(A \cap B) = \frac{1}{5} - \frac{1}{15} = \frac{2}{15}$
$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{4}{5} = \frac{1}{5}$
$P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - \frac{1}{15} = \frac{14}{15}$

13. A problem in statistics is given to three students A, B, C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}\ and\ \frac{1}{4}$respectively. What is the probability that problem is solved?

Sol: A, B and C are events of solving a problem
Given $P(A) = \frac{1}{2},\ P(B) = \frac{3}{4},\ P(C) = \frac{1}{4}$
The required probability $P(A \cup B \cup C) = 1 - [P(A \cup B \cup C)^c]$
$= 1 - P[A^c \cap B^c \cap C^c] = 1 - P(A^c).P(B^c).P(C^c)$
$= 1 - [(1 - P(A))(1 - P(B))(1 - P(C))] = \left[1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right)\right]$
$= \left[1 - \frac{1}{2}.\frac{1}{4}.\frac{3}{4}\right] = \left[1 - \frac{3}{32}\right] = \frac{29}{32}$

14. A and B are events with $P(A) = \frac{1}{3},\ P(B) = \frac{1}{4}\ and\ P(A \cup B) = \frac{1}{2}$.
$Find\ P\left(\frac{A}{B}\right),\ P(A \cup B^c)$

Sol: Given $P(A) = \frac{1}{3},\ P(B) = \frac{1}{4},\ P(A \cup B) = \frac{1}{2}$
$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$
$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$

$P(A \cup B^c) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$

15. $P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(A \cap B) = \frac{1}{5},\ Find\ P(A \cup B),\ P\left(\frac{A^c}{B}\right),\ P\left(\frac{A}{B^c}\right),\ P\left(\frac{A^c}{B^c}\right)$

Sol: Given $P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(A \cap B) = \frac{1}{5}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{5} = \frac{19}{30}$

$P(A^c \cap B) = P(B) - P(A \cap B) = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

$P(A \cap B^c) = P(A) - P(A \cap B) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$

$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{19}{30} = \frac{11}{30}$

16. Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue and 15 orange marbles with replacement being made after each drawing. Find the probability that i) both are white
ii) first is red and second is white

Sol: With replacement events are independent.
Let W_1 is event of drawing white marble in first attempt.
Let W_2 be the event of drawing another white marble in second attempt.
Let R be the event of drawing a red marble

$n(W_1) = n(W_2) = {}^{30}C_1 = 30$

$n(R) = {}^{10}C_1 = 10$

$n(S) = 10 + 30 + 20 + 15 = 75$

i) $P(W_1) = \frac{n(W_1)}{n(S)} = \frac{30}{75} = \frac{2}{5}$

$P(W_2) = \frac{n(W_2)}{n(S)} = \frac{30}{75} = \frac{2}{5}$

$P(W_1 \cap W_2) = P(W_1).P(W_2) = \frac{2}{5}.\frac{2}{5} = \frac{4}{25}$

ii) first is red and second is white:

$P(R) = \frac{n(R)}{n(S)} = \frac{10}{75} = \frac{2}{15}$

$P(W_2) = \frac{2}{5}$

$P(R \cap W_2) = P(R).P(W_2) = \frac{2}{15}.\frac{2}{5} = \frac{4}{75}$

17. In a certain class 25% of the students failed in Maths, 15% failed in Chemistry and 10% in Maths and Chemistry
1) If a student failed in Chemistry, what is the probability that he failed in Maths. $P\left(\frac{B}{A}\right)$
2) If a student failed in Maths, what is the probability that he failed in Chemistry. $P\left(\frac{A}{B}\right)$

Sol: Let $n(S) = 100,\ P(A) = \frac{25}{100},\ P(B) = \frac{15}{100}\ , P(A \cap B) = \frac{10}{100}$
Where A = Maths and B = Chemistry

PRACTICE PROBLEMS:

* Determine $P\left(\frac{B}{A}\right), P\left(\frac{A}{B^c}\right)\ if\ P(A) = \frac{1}{3}, P(B) = \frac{1}{4},\ P(A \cup B) = \frac{1}{2}$

* A bag contains 8 red, 6 blue balls. Two drawing of each two balls are made. Find the probability that first drawing gives two red balls and second drawing gives two blue balls. If the balls are drawn replaced before second drawn.

TOTAL PROBABILITY:

If B_1, B_2, …..B_n are n mutually exclusive events such that $\bigcup_{n=1}^{N} B_n = S,\ P(B_n) > 0\ for\ n = 1,2,3, \dots \dots\ \infty$ then for any arbitrary event A defined on S

$P(A) = \sum_{n-1}^{N} P(A \cap B_n) = \sum_{n=1}^{N} P(B_n).P\left(\frac{A}{B_n}\right)$

$N = Total\ No. of\ elements\ in\ S$

Proof: $\bigcup_{n=1}^{N} B_n = S$

$\therefore A \cap S = A$

$A = A \cap S = A \cap (\bigcup_{n=1}^{N} B_n)$

$A = \bigcup_{n=1}^{N}(A \cap B_n)$

$P(A) = P(\bigcup_{n=1}^{N}(A \cap B_n))$

$= \sum_{n=1}^{N} P(A \cap B_n) \quad A \cap B_1, A \cap B_2, \dots ..are\ mutually\ exclusive$

$P(A) = \sum_{n=1}^{N} P(B_n).P\left(\frac{A}{B_n}\right)$

Total Probability is useful to find $P(A)$ where event A is a particular event of the possible events B_1, B_2, …..B_n.

BAYES THEOREM:

$E_1, E_2 \dots ..E_n$are n mutually exclusive and exhaustive events such that $P(E_i) > 0$ in a sample space. A is any other event in S. Intersection with every E, $P(A) > 0$

$P(E_1),\ P(E_2), \dots ..P(E_n)\ and\ P\left(\frac{A}{E_1}\right),\ P\left(\frac{A}{E_2}\right), \dots ..P\left(\frac{A}{E_n}\right)$ are known then

$$P\left(\frac{E_k}{A}\right) = \frac{P(E_k).P\left(\frac{A}{E_k}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+\cdots..P(E_n).P\left(\frac{A}{E_n}\right)}$$

Proof: $E_1,\ E_2,\ \dots \dots\ E_n$ are n events of S such that $P(E_i) > 0\ and\ E_i \cap E_j = \emptyset\ for\ i\ \neq\ j$ $where\ i,j = 1,2,3 \dots .n$

$E_1, E_2 \dots ..E_n$are exhaustive events.

$S = E_1 \cup E_2 \dots \dots \cup E_n$

$A = A \cap S = A \cap (E_1 \cup E_2 \cup \dots .E_n)$

$A = (A \cap E_1) \cup (A \cap E_2) \cup \dots ..(A \cap E_n)$

$P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup \dots ..(A \cap E_n)]$

$(A \cap E_1), (A \cap E_2), \dots ..(A \cap E_n)$are mutually exclusive events

$P(A) = P(A \cap E_1) + P(A \cap E_2) + \cdots ..P(A \cap E_n)$

$P(A) = P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right) + \cdots + P(E_n).P\left(\frac{A}{E_n}\right)$

Let $E_k\ be\ any\ event$

$$P\left(\frac{E_k}{A}\right) = \frac{P(E_k \cap A)}{P(A)} = \frac{P(E_k).P\left(\frac{A}{E_k}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+\cdots..P(E_n).P\left(\frac{A}{E_n}\right)}$$

PROBLEMS:

1. Three students A, B, C are in a running race . A and B have same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins

Sol: Given $P(A) = 2P(C)$

$P(B) = 2P(C)$

But $P(A) + P(B) + P(C) = 1$

$2P(C) + 2P(C) + P(C) = 1 \Rightarrow 5P(C) = 1$

$\Rightarrow P(C) = \frac{1}{5} \Rightarrow P(B) = \frac{2}{5},\ P(A) = \frac{2}{5}$

$P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$= \frac{2}{5} + \frac{1}{5} - P(B).P(C) = \frac{2}{5} + \frac{1}{5} - \frac{2}{25} = \frac{13}{25}$

PRACTICE PROBLEM:

2. A bag contain 4 green, 6 black, 7 white balls. A ball is drawn at random. What is the probability that it is either green or black ball, black or white.

3. A bag contains 60 balls numbered 1 to 60. One ball is drawn at random, find the probability that the number on the ball drawn will be a multiple of 3 or 7

Sol: $n(A) = \frac{60}{3} = 20$; $P(A) = \frac{20}{60}$

$n(B) = \frac{60}{7} = 8$; $P(B) = \frac{8}{60}$

$n(A \cap B) = \frac{60}{21} = 2$; $P(A \cap B) = \frac{2}{60}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{20}{60} + \frac{8}{60} - \frac{2}{60} = \frac{26}{60}$

4. A, B, C are aiming to shoot a balloon will succeed 4 times out of 5 attempts. The chance of B shoot the balloon is 3 out of 4 and that of C is 2 out of 3. If three aim balloon simultaneously find the probability at least two of them hit the balloon.

Sol: $P(A) = \frac{4}{5},\ P(B) = \frac{3}{4}, P(C) = \frac{2}{3}$

The probability of atleast two i.e., either (A, B), (B, C), (C, A) or all three shooting the balloon.

$= P[A \cap B \cap \bar{C}] \cup [\bar{A} \cap B \cap C] \cup [A \cap \bar{B} \cap C] \cup [A \cap B \cap C]$ (these are mutually exclusive)

$= P[A \cap B \cap \bar{C}] + P[\bar{A} \cap B \cap C] + P[A \cap \bar{B} \cap C] + P[A \cap B \cap C]$

$= P(A).P(B).P(\bar{C}) + P(\bar{A}).P(B).P(C) + P(A).P(\bar{B})P(C) + P(A).P(B).P(C) = \frac{5}{6}$

PRACTICE PROBLEM:

5. A class contain 6 girls and 10 boys. If a committee of 3 is chosen at random from the class, find the probability that
1) 3 boys are selected and 2) exactly two girls are selected.

PROBLEMS ON BAYE'S THEOREM:

A businessman goes to hotels X, Y, Z with 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbing what is the probability that business man room having faulty plumbing is assigned to hotel Z.

Sol: A, B, C are the events businessman chooses hotel X, Y, Z respectively.

f be the event of room faulty plumbing.

$P(A) = \frac{20}{100},\ P(B) = \frac{50}{100},\ P(C) = \frac{30}{100}$

$P\left(\frac{f}{A}\right) = \frac{5}{100}$ $(faulty\ plumbing\ room\ that\ A\ chooses)$

$P\left(\frac{f}{B}\right) = \frac{4}{100}$

$P\left(\frac{f}{C}\right) = \frac{8}{100}$

By the Total Probability theorem

$P(f) = \sum_{i=1}^{3} P(E_i \cap f) = P(E_1 \cap f). + P(E_2 \cap f) + P(E_3 \cap f)$

$= P(E_1).P\left(\frac{f}{E_1}\right) + P(E_2).P\left(\frac{f}{E_2}\right) + P(E_3).P\left(\frac{f}{E_3}\right) = \frac{540}{10000} = \frac{27}{500}$

Probability that hotel Z was selected

$P\left(\frac{E_3}{f}\right) = \frac{P(E_3).P\left(\frac{f}{E_3}\right)}{P(f)} = \frac{\frac{30}{100}.\frac{8}{100}}{\frac{540}{10000}} = \frac{240}{5000} = \frac{4}{9}$

1. In a bolt factory machines A, B, C manufacture 20%, 30% and 50% of the total of their output and 6%, 3% and 2% are defective for 3 machines A, B, C. A bolt is drawn at random and found to be defective. Find the probabilities that is manufactured from
i) Machine A, ii) Machine B, iii) Machine C

Sol: Let P(A), P(B), P(C) denote the probabilities of events bolts are manufactured by the machines A, B, C

Given $P(A) = \frac{20}{100} = \frac{1}{5},\ P(B) = \frac{30}{100} = \frac{3}{10},\ P(C) = \frac{50}{100} = \frac{1}{2}$

Let D is the event of selected bolt is defective

$P\left(\frac{D}{A}\right) = \frac{6}{100} = \frac{3}{50},\ P\left(\frac{D}{B}\right) = \frac{3}{100},\ P\left(\frac{D}{C}\right) = \frac{2}{100} = \frac{1}{50}$

i) If the bolt is defective probability that is from Machine A

$$P\left(\frac{A}{D}\right) = \frac{P(A)P\left(\frac{D}{A}\right)}{P(A).P\left(\frac{D}{A}\right)+P(B).P\left(\frac{D}{B}\right)+P(C).P\left(\frac{D}{C}\right)} = \frac{12}{31}$$

Similarly, ii) $P\left(\frac{B}{D}\right) = \frac{9}{31}$

iii) $P\left(\frac{C}{D}\right) = \frac{10}{31}$

2. First box contains 2 black, 3 red, 1 white ball, second box contains 1 black, 1 red and 2 white balls, third box contains 5 black, 3 red, 4 white balls. Of these a box is selected at random from it a red ball is randomly drawn. If the ball is red find the probability that is from second box.

Sol: $P(B_1) = \frac{1}{3},\ P(B_2) = \frac{1}{3},\ P(B_3) = \frac{1}{3},$

$P\left(\frac{R}{B_1}\right) = \frac{3}{6},\ P\left(\frac{R}{B_2}\right) = \frac{1}{4},\ P\left(\frac{R}{B_3}\right) = \frac{3}{12} = \frac{1}{4}$

$$P\left(\frac{B_2}{R}\right) = \frac{P(B_2)P\left(\frac{R}{B_2}\right)}{P(B_1).P\left(\frac{R}{B_1}\right)+P(B_2).P\left(\frac{R}{B_2}\right)} = \frac{\frac{1}{3}.\frac{1}{4}}{\frac{1}{3}.\frac{3}{6}+\frac{1}{3}.\frac{1}{4}+\frac{1}{3}.\frac{1}{4}} = \frac{1}{4}$$

RANDOM VARIABLE:

FUNCTION:

Let S and T be sets. Suppose for each s ∈ S there is assigned a unique element of T. The collection of such assignments is called a function (or mapping) from S into T and is written as $f : S \rightarrow T$, where S is domain, T is co-domain, f(s) is Range.

RANDOM VARIABLE:

A random variable X on a sample space S is a function from S into the set R of Real numbers, such that the pre image of every element of R is an event of S.

X is a random variable $\rightarrow X : S \rightarrow R$

Example: In tossing a coin two times S = {HH, HT, TH, TT}

Assign each element of S to a real number line

X: S→R X(S) = No. of heads

$X(TT) = 0, X(HH) = 2, X(HT) = 1, X(TH) = 1$

$P(TT) = \frac{1}{4},\ P(TH, HT) = \frac{1}{2}\ ,\ P(HH) = \frac{1}{4}$

X	0	1	2
P(x)	¼	½	¼

A function to become a random variable it must satisfy the following conditions

1) $P(X \leq x)$is an event for x

2) $P(X = \infty) = P(X = -\infty) = 0$

DISCRETE RANDOM VARIABLE:

A random variable which can take only a finite number is called discrete random variable. Every real valued function in sample space is random variable.
Example :$X(s) = \{S: s = 0, 1, 2\}$

CONTINUOUS RANDOM VARIABLE:

A random variable X is said to be continuous if it take all possible values in an interval. A continuous random variable is a random variable that can be measured any desired degree of accuracy. Example: age, height, weight.

* Height of a student in a particular class may be between 4 ft and 6 ft.
$X(s) = \{x : 4 \leq x \leq 6\}$

PROBABILITY MASS FUNCTION (OR) PROBABILITY FUNCTION:

X be a discrete random variable. The discrete probability function f(x) for X is given by $f(x) = P(X = x)$ for all real x.

PROPERTIES:

$f(x) = \frac{d}{dx}[F(x)]$
1. $f(x) \geq 0\ \forall\ x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(E) = \int_E f(x)dx$is well defined for any event E.

DISCRETE PROBABILITY DISTRIBUTION:

Probability distribution of tossing two coins

X	0	1	2
P(x)	¼	½	¼

when X(s) = No. of heads.

PROBABILITY DENSITY FUNCTION: Derivative of Distribution Function is density function.
The function f(x) so defined is known as Probability density function or simply density function of random variable X. Derivative of commutative distribution function.

* 1) The expression f(x)dx is known as the probability differential.

* 2) The curve y = f(x) is known as the probability density curve or simple probability curve.

MEAN, VARIANCE FOR DISCRETE PROBABILITY DISTRIBUTION:

The Mean of probability distribution is simply mathematical expectation of a corresponding random variable if random variable X takes values $x_1, x_2, \ldots\ldots x_n$ with probabilities $f(x_1), f(x_2), \ldots\ldots f(x_n)$. The mathematical expectation or expected value is
$x_1 f(x_1) + x_2 f(x_2) + \cdots x_n f(x_n) = Mean$
$Mean\ (\mu) = \sum_{i=1}^{n} x_i P(x)$
$Variance\ (\sigma^2) = \sum_{i=1}^{n} x_i^2 P(x_i) - \mu^2 = E(X^2) - [E(X)]^2$

PROBLEMS :

1. X and Y are discrete random variable and k is constant then prove that
i) $E(X + k) = E(X) + k$
ii) $E(X + Y) = E(X) + E(Y)$

Proof: By definition we have

i) $E(X+k)=\sum_{i=1}^{n}(x_i+k)P_i = \sum_{i=1}^{n} x_iP_i + k\sum_{i=1}^{n} P_i$

$E(X+k)=E(X)+k \quad [\sum_{i=1}^{n} P_i = 1]$

ii) $E(X+Y)=E(X)+E(Y)$

$E(X)=\sum_{i=1}^{n} P_ix_i \quad ; \quad E(Y)=\sum_{j=1}^{m} P_jy_j$

$E(X+Y)=\sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}\left(x_i+y_j\right)$

$=\sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}x_i+\sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij}y_j$

$=\sum_{i=1}^{n}\left[x_i\sum_{j=1}^{m} P_{ij}\right]+\sum_{j=1}^{m}\left[y_i\sum_{i=1}^{n} P_{ij}\right]$

$=\sum_{i=1}^{n} x_iP_i+\sum_{j=1}^{m} y_jP_j$

$E(X+Y)=E(X)+E(Y)$

PROBLEMS ON DISCRETE RANDOM VARIABLE:

* Let X denote the minimum of two numbers that appear when a pair of fair dice thrown once. Determine the i) Discrete probability ii) Expectation iii) Variance

Sol: $P(X=1)=P\{(1,1),(1,2),(2,1),(1,3),(3,1),(4,1),(1,4),(5,1),(1,5),(6,1),(1,6)\}=\frac{11}{36}$

$P(X=2)=P\{(2,2),(2,3),(3,2),(2,4),(4,2),(2,5),(5,2),(2,6),(6,2)\}=\frac{9}{36}$

$P(X=3)=P\{(3,3),(3,4),(4,3),(3,5),(5,3),(3,6),(6,3)\}=\frac{7}{36}$

$P(X=4)=P\{(4,4),(4,5),(5,4),(4,6),(6,4)\}=\frac{5}{36}$

$P(X=5)=P\{(5,5),(5,6),(6,5)\}=\frac{3}{36}$

$P(X=6)=P\{(6,6)\}=\frac{1}{36}$

Probability Distribution:

X	1	2	3	4	5	6
P(x)	11/36	9/36	7/36	5/36	3/36	1/36

Mean $=\sum xP(x)=1\times\frac{11}{36}+2\times\frac{9}{36}+3\times\frac{7}{36}+4\times\frac{5}{36}+5\times\frac{3}{36}+6\times\frac{1}{36}=\frac{91}{36}=2.52$

Variance $=\sum x^2P(x)-\mu^2$

$=1\times\frac{11}{36}+4\times\frac{9}{36}+9\times\frac{7}{36}+16\times\frac{5}{36}+25\times\frac{3}{36}+36\times\frac{1}{36}-\left(\frac{91}{36}\right)^2$

$=\frac{301}{36}-\left(\frac{91}{36}\right)^2=1.97$

1. Let X denote the number of heads in a single toss of 4 fair coins. Determine
   i) $P(X<2)$ ii) $P(1<X\le 3)$

Sol: Given that four fair coins are tossed . Sample space $S=2^4=16$

$$S=\begin{Bmatrix}(HHHH) & (HTTT) & (THHH) & (HTHT)\\ (THTH) & (HTHH) & (THTT) & (HHTT)\\ (HHHT) & (TTTH) & (TTHH) & (THHT)\\ (TTHT) & (HHTH) & (HTTH) & (TTTT)\end{Bmatrix}$$

Let X denote the No. of heads

$P(X=0)=\frac{1}{16}$ $\mathit{Probability\ of\ getting\ no\ heads}$

$P(X=1)=\frac{4}{16}$

$P(X=2)=\frac{6}{16}$

$P(X=3)=\frac{4}{16}$

$P(X=4)=\frac{1}{16}$

X	0	1	2	3	4
P(x)	1/16	4/16	6/16	4/16	1/16

i) $P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$

ii) $P(1 < X \leq 3) = P(X = 2) + P(X = 3) = \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}$

2. Two dice are thrown. Let X be the random variable assign to each point (a, b) in S the maximum of its number. Find the distribution, the mean and variance of the distribution.

Sol: X is a random variable

$P(X = 1)(1\ is\ maximum) = P(1, 1) = \frac{1}{36}$

$P(X = 2) = P[(2, 1), (1, 2), (2, 2)] = \frac{3}{36}$

$P(X = 3) = P[(1, 3), (3, 1), (2, 3), (3, 2), (3, 3)] = \frac{5}{36}$

$P(X = 4) = P[(1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4)] = \frac{7}{36}$

$P(X = 5) = P[(1, 5), (5, 1), (2, 5), (5, 2), (3, 5), (5, 3), (4, 5), (5, 4), (5, 5)] = \frac{9}{36}$

$P(X = 6) = \frac{11}{36}$

X	1	2	3	4	5	6
P(x)	1/36	3/36	5/36	7/36	936	11/36

Mean= $\sum_{i=1}^{n} x_i P(x_i) = 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36} = 4.47$

Variance $= \sum_{i=1}^{n} x_i{}^2 P(x_i) - \mu^2 = 1.99 \cong 2$

EXERCISE :

Two dice are thrown. Let X be the random variable assign to each point (a, b) in S the minimum of its number. Find the distribution mean and variance

3. A random variable X has the following probability function

X	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

i) k ii) $P(X < 6)$ iii) $P(X \geq 6)$ iv) $P(0 < X < 5)$ v) Mean

vi) Variance vii) Smallest values of X such that $P(X \leq x) > \frac{1}{2}$

Sol: Sum of probability = 1

i) $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1 \Rightarrow 10k^2 + 9k - 1 = 0$

$\Rightarrow 10k^2 + 10k - k - 1 = 0 \Rightarrow 10k(k + 1) - 1(k + 1) = 0$

$\Rightarrow (10k - 1)(k + 1) = 0 \Rightarrow 10k - 1 = 0; k + 1 = 0 \Rightarrow k = \frac{1}{10}, -1(negative)$ $\boxed{K = \frac{1}{10}}$

ii) $P(X < 6) = P(X = 0) + P(X = 1) + \cdots + P(X = 5)$

$= 0 + k + 2k + 2k + 3k + k^2 = k^2 + 8k = 0.81$

iii) $P(X \geq 6) = P(X = 6) + P(X = 7)\ \ or\ 1 - P(X < 6) = 1 - 0.81 = 0.19$

iv) $P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= k + 2k + 2k + 3k = 8k = \frac{8}{10} = 0.8$

v) Mean= $\sum xP(x) = 66k^2 + 30k = 66 \times \left(\frac{1}{10}\right)^2 + 30 \times \frac{1}{10} = \frac{66}{100} + \frac{30}{10} = \frac{366}{100} = 3.66$

vi) Variance= $E(X^2) - [E(X)]^2 = \sum X^2 P(x) - \mu^2 = 440k^2 + 124k = 16.8$

vii) Smallest value of X is determined by trial method

$P(X \leq 0) = 0$

$P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{10}$

$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{3}{10}$

$P(X \leq 3) = \frac{5}{10}$

$P(X \leq 4) = \frac{8}{10} > \frac{1}{2}$

∴ Smallest values of X for which $P(X \leq x) > \frac{1}{2}\ is\ 4$

EXERCISE :

Calculate Mean.

X =x	0.3	0.2	0.1	0	1	2	3
P(X=x)	0.05	0.10	0.30	0	0.30	0.15	0.1

Ans: $Mean = 0.965$

*

X	1	2	3	4	5	6	7	8
P(x)	k	2k	3k	4k	5k	6k	7k	8k

Find $k, P(X \leq 2), P(2 \leq X \leq 5)$

Ans: $k = \frac{1}{36}$; $P(X \leq 2) = \frac{1}{12}$

$P(2 \leq X \leq 5) = \frac{7}{18}$

4. A player tosses 3 fair coins. He wins Rs. 500 if 3 heads appear, Rs 300 if 2 heads appear, Rs 100 if 1 head occur. On other hand he losses 1500 if 3 tails occur find the expected gain of the player.

Sol: Let X be the expected gain of player

The range of X is {-1500, 100, 300, 500}

Let S be the sample space of tossing 3 fair coins i.e.,$n(S) = 2^3 = 8$

$S = \{HHH, HHT, HTH, HTT, THH, TTH, TTT\}$

Let A be the event of getting a_{11} heads

Probability of winning is Rs 500. $P(X = 3) = \frac{1}{8}$

B be the event of getting 2 heads $P(B) = \frac{3}{8}$

C be the event of getting one head $P(C) = \frac{3}{8}$

D be the event of getting no heads $P(D) = \frac{1}{8}$

X	-1500	100	300	500
P(X)	1/8	3/8	3/8	1/8

$E(X) = \sum xP(X) = Rs\ 25$

5. A bag contains 10 items. A man draws 3 items from bag. Find expected number of defective item by draw. (Assume there are 3 defective)

Sol: Number of defective = 3

Total number of items = 10

Number of good items = 7

Probability of getting 0 defective$P(X = 0) = \frac{^7C_3}{^{10}C_3} = \frac{35}{120} = \frac{7}{24} = 0.29$

1 defective $P(X = 1) = \frac{^3C_1 \times {^7C_2}}{^{10}C_3} = \frac{21}{40} = 0.525$

2 defective $P(X = 2) = \frac{^3C_2 \times {^7C_1}}{^{10}C_3} = \frac{7}{40} = 0.175$

3 defective$P(X = 3) = \frac{1}{^{10}C_3} = \frac{1}{120} = 0.0083$

X	0	1	2
P(x)	35/120	21/40	7/40

$E(x) = \sum_{x=0}^{2} xP(x) = 0 \times \frac{35}{120} + 1 \times \frac{21}{40} + 2 \times \frac{7}{40} \Rightarrow \quad E(x) = 0.89$

FOR CONTINUOUS RANDOM DISTRIBUTION:

1. DISTRIBUTION FUNCTION:

$f_x(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \quad -\infty < x < \infty$

PROPERTIES:

1. $0 \leq F(x) \leq 1$
2. $F(-\infty) = 0, F(\infty) = 1$
3. $P(a \leq x \leq b) = \int_a^b f(x)dx = f(b) - f(a)$

2. DENSITY FUNCTION PROPERTIES:

Derivative of distribution function is density function $f(x) = \frac{d}{dx}[F(x)]$

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(E) = \int_E f(x)dx$

3. MEAN OF DISTRIBUTION :

$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$

X defined from a to b then

$\mu = E(X) = \int_a^b f(x)dx$

Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_a^b x^2 f(x) - \mu^2$

4. MEDIAN:

Median is point which divides the entire distribution into two equal parts. In case of continuous distribution median is point which divides the total area into two equal parts. Thus if X is defined from a to M is the median, then

$\int_a^M f(x)dx \; = \int_M^b f(x)dx = \frac{1}{2}$

Solving for M we get Median.

5. MODE :

Mode is value of x for which f(x) is maximum. Mode is thus given $f'(x) = 0 \; and \; f''(x) < 0$

$for \; a < x < b.$

1. F(x) be the distribution function of a random variable given by

$$F(x) = \begin{cases} cx^3 \; when \; 0 \leq x < 3 \\ \quad 1 \; when \; x \geq 3 \\ \quad 0 \; when \; x < 0 \end{cases}$$

Determine i) c , ii) density function, iii) mean, iv) $P(X > 1)$

Sol: Given $F(x) = cx^3$

i) $F(x)$is distributive function if f(x) is density fuction

$f(x) = \frac{d}{dx}[cx^3] = 3x^2c$

We know from density function property

$\int_{-\infty}^{\infty} f(x)dx = 1$

$\int_0^3 f(x)dx = 1 \Rightarrow \int_0^3 c3x^2dx = 1 \Rightarrow 3c\int_0^3 x^2dx = 1 \Rightarrow 3c\left[\frac{x^3}{3}\right]_0^3 = 1$

$\Rightarrow c[3^3 - 0] = 1 \Rightarrow c = \frac{1}{27}$

ii) Probability density function:

$f(x) = \frac{d}{dx}[f(x)] = c3(x^3)$

$$f(x) = \begin{cases} c.3x^2 & 0 \le x < 3 \\ 0 & for\ x \ge 3 \\ 0 & for\ x < 0 \end{cases}$$

iii) Mean $= \int_0^3 x\,.3cx^2dx = 3c\int_0^3 x^3dx = 3\left[\frac{x^4}{4}\right]_0^3 = \frac{3}{27}\left[\frac{3^4}{4}\right] = \frac{81}{36} = \frac{9}{4}$

iv)$P(X > 1) = 3c\int_1^3 x^2dx = 3 \times \frac{1}{27}\left[\frac{x^3}{3}\right]_1^3 = \frac{1}{27}[27 - 1] = \frac{26}{27}$

2. If a random variable has the probability density $f(x) = \begin{cases} 2e^{-2x} & for\ x > 0 \\ 0 & for\ x \le 0 \end{cases}$. Find the probabilities that it will take on a value.
i) between 1 and 3 and ii) greater than 0.5

Sol: i) $P(1 \le x \le 3) = \int_1^3 2e^{-2x}dx = \left[2 \times \frac{e^{-2x}}{-2}\right]_1^3 = -[e^{-6} - e^{-2}]$

$= e^{-2} - e^{-6} = 0.1328$

ii) $P(x > 0.5) = \int_{0.5}^{\infty} 2e^{-2x}dx = 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty} = e^{-\infty} + e^{-2\times\frac{1}{2}} = e^{-1} = 0.3678$

3. Let $f(x) = 3x^2\ when\ 0 \le x \le 1$ be the probability density function of a continuous random variable X. Determine a and b such that
i) $P(X \le a) = P(X > a)$ ii) $P(X > b) = 0.05$

Sol: We know that $\sum P(x) = 1$

$P[X < a] = P[x \le a] = \frac{1}{2}$

$\int_0^a f(x)dx = \frac{1}{2} \Rightarrow \int_0^a 3x^2dx = \frac{1}{2} \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = \frac{1}{2} \Rightarrow a^3 = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$

$P(X > b) = 0.05$

$\int_b^1 f(x)dx = 0.05 \Rightarrow \int_b^1 3x^2dx = 0.05 \Rightarrow [x^3]_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05 - \frac{1}{20}$

$\Rightarrow b^3 = 1 - \frac{1}{20} \Rightarrow b^3 = \frac{19}{20} \Rightarrow b = \left(\frac{19}{20}\right)^{\frac{1}{3}} = 0.9830$

4. A continuous random variable X has the distribution function $F(x) =$

$$\begin{cases} 0 & if\ x \le 1 \\ k(x-1)^4 & if\ 1 < x \le 3 \\ = 1 & if\ x > 3 \end{cases}$$

Find k and the probability density function.

Sol: $f(x) = \frac{d}{dx}[f(x)]$

$$f(x) = \begin{cases} 0 & if\ x \le 1 \\ 4k(x-1)^3 & 1 \le x \le 3 \\ 0 & if\ x > 3 \end{cases}$$

We have $\int_{-\infty}^{\infty} f(x)dx = 1$

$\int_{-\infty}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$

$\Rightarrow 0 + 4k\int_{1}^{3}(x-1)^3 dx + 0 = 1 \Rightarrow 4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1 \Rightarrow 4k\left[\frac{2^4}{4} - 0\right] = 1 \Rightarrow k = \frac{1}{16}$

5. The probability density function $f(x) = \begin{cases} kx^2 \ in \ 1 \le x \le 3 \\ 0 \ elsewhere \end{cases}$

Find the value of k and find the probability between 1 and 3/2.

Sol: Density function

$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$

$\int_{1}^{3} f(x)dx = 1 \Rightarrow \int_{1}^{3} kx^3 dx = 1 \Rightarrow k\left[\frac{x^4}{4}\right]_1^3 = 1 \Rightarrow \frac{k}{4}[3^4 - 1] = 1 \Rightarrow \frac{k}{4}[81-1] = 1$

$\Rightarrow k = \frac{1}{20}$

$P\left[1 \le x \le \frac{3}{2}\right] = \int_{1}^{\frac{3}{2}} f(x)dx = \frac{3}{26}$

6. Let $f(x) = 3x^2 \ where \ 0 \le x < 1$ the probability density function of continuous variable X. Determine a and b such that

i) $P(X \le a) = P(X > a)$

ii) $P(X > b) = 0.05$

Sol: We know that $\sum P(x) = 1$

i) $P(X \le a) = P(X > a) = \frac{1}{2}$

$P(X \le a) = \frac{1}{2}$

$\int_{0}^{a} f(x)dx = \frac{1}{2} \Rightarrow \int_{0}^{a} 3x^2 dx = \frac{1}{2} \Rightarrow [x^3]_0^a = \frac{1}{2} \Rightarrow a^3 = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$

ii) $P(X > b) = 0.05 \Rightarrow \int_{b}^{1} f(x)dx = 0.05 \Rightarrow \int_{b}^{1} 3x^2 dx = 0.05 \Rightarrow 1 - b^3 = \frac{1}{20}$

$\Rightarrow b^3 = \frac{19}{20} \Rightarrow b = \left(\frac{19}{20}\right)^{\frac{1}{3}} = 0.793$

7. $f(x) = ke^{-|x|}$is a probability density function in $-\infty < x < \infty$. Find the value of k, variance, probability between 0 and 4.

Sol: Value of k

$\int_{-\infty}^{\infty} f(x)dx = k \Rightarrow \int_{-\infty}^{0} ke^{-|x|}dx + \int_{0}^{\infty} ke^{-|x|}dx = 1$

$[in - \infty \ to \ 0 \ |x| = -x \ \ and \ 0 \ to \ \infty \ |x| = x]$

$k\int_{-\infty}^{0} e^{x}dx + k\int_{0}^{\infty} e^{-x}dx = 1 \Rightarrow k[e^x]_{-\infty}^{0} + k[e^{-x}]_0^{\infty} = 1 \Rightarrow k[1+0] + k[1] = 1$

$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$

Variance$\sigma^2 = E(X^2) - [E(X)]^2$

$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} xke^{-|x|}dx = \frac{1}{2}\int_{-\infty}^{0} xe^{-|x|}dx + \frac{1}{2}\int_{0}^{\infty} xe^{-|x|}dx$

$= \frac{1}{2}\int_{-\infty}^{0} xe^{x}dx + \frac{1}{2}\int_{0}^{\infty} xe^{-x}dx = \frac{1}{2}[e^x(x-1)]_{-\infty}^{0} + [e^{-x}(-x-1)]_0^{\infty}$

$= \frac{1}{2}\left[(-1-0) + \left(0-(-1)\right)\right] = \frac{1}{2}[-1+1] = 0$

$\Rightarrow E(X) = 0$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{0} x^2 e^{-|x|}dx + \int_{0}^{\infty} x^2 e^{-|x|}dx = \frac{1}{2}\int_{-\infty}^{0} x^2 e^{x}dx + \frac{1}{2}\int_{0}^{\infty} x^2 e^{-x}dx$

$\frac{1}{2}\int_{-\infty}^{0} x^2 e^{x}dx = \frac{1}{2}\left[x^2\int e^x dx - \int\left(\frac{d}{dx}x^2 \int e^x dx\right)dx\right]_{-\infty}^{0} = \frac{1}{2}[x^2 e^x - 2\int xe^x dx]_{-\infty}^{0}$

$= \frac{1}{2}[x^2e^x - 2e^x(x-1)]_{-\infty}^{0} = \frac{1}{2}[0-(-2)-0] = \frac{1}{2}\times 2 = 1$ $\boxed{e^{-\infty}=0}$

$\frac{1}{2}\int_0^\infty x^2 e^{-|x|}dx = \frac{1}{2}\int x^2e^{-x}dx = \frac{1}{2}[-x^2e^{-x} + 2\int(xe^{-x}dx)dx]_0^\infty$ $\boxed{e^0=1}$

$= \frac{1}{2}[-x^2e^{-x} + 2e^{-x}(-x-1)]_0^\infty = \frac{1}{2}[0-(0-2)] = \frac{1}{2}\times 2 = 1$

$E[X^2] = 1+1 = 2$

∴ Variance $= E(X^2) - [E(X)]^2 = 2 - 0 = 2$

Probability between 0 and 4

$P(0\le x\le 4) = \frac{1}{2}\int_0^4 e^{-|x|}dx = \frac{1}{2}[-e^{-x}]_0^4 = \frac{1}{2}[-(e^{-4}-e^{-0})] = 0.0183 - 1 = 0.49$

8. X is continuous random variable w.r.to density distribution $f(x) = \begin{cases}\frac{1}{6}x + k \ \ if\ 0\le x\le 3\\ \quad 0,\ \ elsewhere\end{cases}$.

Determine i) the value of k, ii) the mean, iii) $P(1\le x\le 2)$

Sol: Given $f(x) = \frac{1}{6}x + k\ ,\ \ 0\le x\le 3$

$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow\ \int_0^3 f(x)dx = 1 \Rightarrow\ \int_0^3 \frac{1}{6}x + k\ dx = 1 \Rightarrow \frac{1}{6}\int_0^3 xdx + k\int dx = 1$

$\Rightarrow \frac{1}{6}\left[\frac{x^2}{2}\right]_0^3 + k[x]_0^3 = 1 \Rightarrow \frac{1}{6}\left[\frac{9}{2}\right] + 3k = 1 \Rightarrow \frac{3}{4} + 3k = 1 \Rightarrow k = \frac{1}{12}$

Mean $= \int_0^3 xf(x)dx = \int_0^3 x\left(\frac{x}{6}+k\right)dx = \frac{1}{6}\int_0^3 x^2dx + k\int_0^3 xdx = \frac{1}{6}\left[\frac{x^3}{3}\right]_0^3 + k\left[\frac{x^2}{2}\right]_0^3$

$= \frac{1}{6}\left[\frac{27}{3}\right] + \frac{1}{12}\left[\frac{3^2}{2}\right] = \frac{3}{2} + \frac{3}{8} = \frac{15}{8} = 1.875$

$P(1\le x\le 2) = \int_1^2 f(x)dx = \int_1^2\left(\frac{x}{6}+k\right)dx = \frac{1}{6}\int_1^2 xdx + k\int_1^2 1dx = \frac{1}{6}\left[\frac{x^2}{2}\right]_1^2 + k[x]_1^2$

$= \frac{1}{6}\left[\frac{4}{2}-\frac{1}{2}\right] + k[2-1] = \frac{1}{6}\left(\frac{3}{2}\right) + k = \frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$

EXERCISE:

1. Probability density function $f(x) = \begin{cases}kx^3\ in\ 1\le x\le 3\\ \ 0\ \ elsewhere\end{cases}$.

Find k probability between 1 and 3/2.

2. Probability density function of X is given by

$$f(x) = \begin{cases}\frac{x}{2} for\ 0 < x\le 1\\ \frac{1}{2}\ for\ 1 < x \le 2\\ \frac{3-x}{2}\ \ for\ 2 < x < 3\\ \ 0\ \ elsewhere\end{cases}$$

Find the expected value of $f(x) = x^2 - 5x + 3$

Hint: $f(x) = \frac{x}{2}\ for\ 0 < x\le 1$

$E(X) = \int_a^b xf(x)dx = \int_0^1 x\frac{x}{2}\ dx = \frac{1}{2}\left[\frac{x^2}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3}-0\right]$

$\Rightarrow E(X) = \frac{1}{6}$

$E(x^2) = \int_0^1 x^2\frac{x}{2}dx\ = \frac{1}{2}\int_0^1 x^3dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{4}-0\right]$

$\Rightarrow E(X^2) = \frac{1}{8}$

Expected value of $f(x) = x^2 - 5x + 3$

$x^2 - 5x + 3 = E(x^2) - 5\big(E(X)\big) + 3 = \frac{1}{8} - 5\times\frac{1}{6} + 3 = \frac{1}{8} - \frac{5}{6} + 3 = \frac{55}{24} = 2.2916$

Continue for $f(x) = \frac{1}{2}\ 0 < x \leq 1$

$E(X) = \frac{3}{4},\ E(X^2) = \frac{7}{6}$

Expected value = $x^2 - 5x + 3 = \frac{7}{6} - 5 \times \frac{3}{4} + 3 = \frac{5}{12}$

$f(x) = \frac{3}{2} - x \quad for\ 2 < x < 3$

$E(X) = \frac{7}{12}, E(X^2) = 11/8$

Expected value = $\frac{35}{24}$

PRACTICE PROBLEMS:

PROBABILITY:

1. Define (i) Sample Space (ii) Mutually exclusive events
2. What is the probability of getting on even number in the throw of a dice
3. If the two outcomes A and B are independent where P(A) = 0.4, P(A∪B)=0.7 then find P(B).
4. A bag contains 12 balls numbered from 1 to 12 If a ball is taken at random, what is the probability of having a ball with a number which is multiple of either 2 or 3.[Ans : 2/3]
5. One card is drawn from a regular deck of 52 cards. What is the probability of card being red or a king. [Ans : $\frac{15}{26}$]
6. A problem in statistics is given to the 3 students A,B,C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem is solved. [Ans : 0.90625]
7. In a bolt factory machine A, B and C manufactures respectively 25% and 35% and 40% of the total of their output 5,4,2 percents are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C ?

DISCRETE PROBABILITY DISTRIBUTION:

1. The probability of a variable X is

(i) Find K, Mean and Variance

X	0	1	2	3	4	5	6
P(X)	K	3K	5K	7K	9K	11K	13K

(ii) Find K, Mean and Variance

X=x	1	2	3	4	5
P(X)	K	2K	3K	4K	5K

2. The probability distribution of a random variable X is given as follows

(i) Find (i) Mean and (ii) Variance

X=x	0	1	2	3	4
P(X=x)	0.4	0.3	0.1	0.1	0.1

3. A variate X has the following probability distribution

X	-3	6	9
P(X)	1/6	1/2	1/3

Find (a) E(X), (b) $E(X^2)$, (c) Variance

CONTINUOUS PROBABILITY DISTRIBUTION:

1. Find the value of k and distribution function F(x) given probability density function of a random variable x as

$f(x) = \frac{K}{x^2+1}, -\infty < x < \infty$

2. The cumulative distribution function for a continuous random variable X is

$$F(x) = \begin{cases} 1-e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find (i) density function f(x) (ii) mean and (ii) variance of the density function.

3. If X is a continuous random variable and Y = ax + b, prove that E(y) = aE(x) + b and V(y) = a^2V(X), where V stands for variance and a,b are constants.

4. A continuous random variable has probability density function

$$f(x) = \begin{cases} Kxe^{-\lambda x}, & x \geq 0, \lambda > 0 \\ 0, & otherwise \end{cases}$$

Determine K, mean, variance.

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